Periods of tropical K3 hypersurfaces

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1. The case of curves

- $K := \overline{\mathbb{C}\{t\}}$: the convergent Puiseux series field
- $f = \sum_i k_i x^i \in K[x_1^{\pm}, x_2^{\pm}]$
- $V(\mathrm{trop}(f))$: the tropical hypersurface defined by $\mathrm{trop}(f)$
- $R \in \mathbb{R}^{>0}$: sufficiently large $\rightsquigarrow f_R := f|_{t=1/R} \in \mathbb{C}[x_1^{\pm}, x_2^{\pm}]$
- $V(R) := \{f_R = 0\}$: the complex hypersurface

Theorem (Katz–Markwig–Markwig, Iwao)

Take holomorphic forms $\{\Omega_i(R)\}_{i=1}^g$ so that $\int_{\alpha_i} \Omega_j(R) = \delta_{i,j}$.
The leading term of the period is given by

$$\int_{\beta_i} \Omega_j(R) \sim \frac{-1}{2\pi\sqrt{-1}} \log R \cdot T_{i,j}, \quad (R \to +\infty)$$

where $\left\{T_{i,j}\right\}_{i,j}$ is the tropical period matrix of $V(\mathrm{trop}(f))$.

2. Tropical K3 surfaces

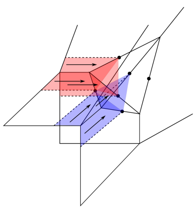
Theorem (Gross–Wilson)

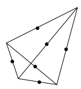
Maximally degenerating families of complex K3 surfaces with Ricci-flat Kähler metrics converge to 2-spheres with integral affine structures with singularities in the Gromov–Hausdorff limit.

The first step : To construct an integral affine 2-sphere from a tropical K3 hypersurface.

3. Contractions

- The projection to the maximal dimensional torus orbit induces an integral affine structure on a neighborhood of a vertex.
- Black dots are singularities.





- Similar/the same constructions have already been performed by Gross–Siebert and Kontsevich–Soibelman.

4. Radiance obstructions

- B : an integral affine manifold (with singularities)
- $\iota \colon B_0 \hookrightarrow B$: the smooth part
- $\mathcal{T}_{\mathbb{Z}}$: the sheaf of integral tangent vectors on B_0
- $\mathcal{T} := \mathcal{T}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{R}$
- $\{U_i\}_i$: a sufficiently fine open covering of B
- $\left\{s_i \in \Gamma(U_i \cap B_0, T^{\mathrm{aff}} B_0)\right\}_i$

Definition (Goldman–Hirsch '84)

The radiance obstruction $c_B \in H^1(B, \iota_* \mathcal{T})$ is defined by

$$c_B((U_i, U_j)) := s_j - s_i$$

for each 1-simplex (U_i, U_j) of $\{U_i\}_i$.

5. Main results

- $\Delta \subset M_{\mathbb{R}}$: a smooth reflexive polytope
- $\check{\Delta} \subset N_{\mathbb{R}}$: the polar polytope of Δ
- $F = \max_{n \in \check{\Delta} \cap N} \{a(n) + n \cdot X\}$: a tropical polynomial
- $V(F)$: the tropical hypersurface defined by F
- B : an integral affine 2-sphere obtained by contracting $V(F)$
- $\Sigma \subset M_{\mathbb{R}}$: the normal fan of Δ
- X_Σ : the complex toric manifold associated with Σ
- D_ρ : the toric divisor on X_Σ corresponding to $\rho \in \Sigma(1)$
- $Y \subset X_\Sigma$: an anti-canonical hypersurface
- $\mathrm{Pic}(Y)_{\mathrm{amb}} := \mathrm{Im}\,\{\mathrm{Pic}(X_\Sigma) \hookrightarrow \mathrm{Pic}(Y)\}$
- $\cup \colon H^1(B, \iota_* \mathcal{T}_{\mathbb{Z}}) \otimes H^1(B, \iota_* \mathcal{T}_{\mathbb{Z}}) \to H^2(B, \iota_* \wedge^2 \mathcal{T}_{\mathbb{Z}}) \cong \mathbb{Z}$

Theorem (Y.)

1. There is a primitive embedding
$$\psi \colon \mathrm{Pic}(Y)_{\mathrm{amb}} \hookrightarrow H^1(B, \iota_* \mathcal{T}_{\mathbb{Z}})$$
that preserves the pairing.
2. The radiance obstruction c_B is given by
$$c_B = \sum_{\rho \in \Sigma(1)} \left\{a(n_\rho) - a(0)\right\} \psi(D_\rho).$$

6. Corollary

- $f = \sum_i k_i x^i \in K[x_1^{\pm}, x_2^{\pm}, x_3^{\pm}]$
- $R \in \mathbb{R}^{>0}$: sufficiently large $\rightsquigarrow f_R := f|_{t=1/R} \in \mathbb{C}[x_1^{\pm}, x_2^{\pm}, x_3^{\pm}]$
- $V(R) := \{f_R = 0\}$: the complex K3 hypersurface
- The period map
$$\mathcal{P} \colon (R_0, \infty) \to \left\{[\sigma] \in \mathbb{P}((U \oplus \mathrm{Pic}(Y)_{\mathrm{amb}}) \otimes \mathbb{C}) | (\sigma, \sigma) = 0, (\sigma, \overline{\sigma}) > 0\right\}$$
$$\cong \{\sigma \in \mathrm{Pic}(Y)_{\mathrm{amb}} \otimes \mathbb{C} | (\Re e \sigma, \Re e \sigma) > 0\}$$
- B : an integral affine 2-sphere obtained by contracting the tropical hypersurface $V(\mathrm{trop}(f))$

Corollary (Y.)

1. The leading term of the period map $\mathcal{P}(R)$ is given by
$$\mathcal{P}(R) \sim \log R \cdot \psi^{-1}(c_B) \quad (R \to +\infty).$$
The element $\psi^{-1}(c_B) \in \mathrm{Pic}(Y)_{\mathrm{amb}} \otimes_{\mathbb{Z}} \mathbb{R}$ can be regarded as the tropical period of $B \simeq V(\mathrm{trop}(f))$.
2. $$(\psi^{-1}(c_B), \psi^{-1}(c_B)) > 0.$$
This can be regarded as the tropical version of Hodge–Riemann bilinear relation.